

Witten-Helffer-Sjöstrand Theory for a Generalized Morse Function

Hon-kit Wai

February 7, 2008

Abstract

In this paper, we extend the Witten-Helffer-Sjöstrand theory from Morse functions to generalized Morse functions. In this case, the spectrum of the Witten deformed Laplacian $\Delta(t)$, for large t , can be separated into the small eigenvalues (which tend to 0 as $t \rightarrow \infty$), large and very large eigenvalues (both of which tend to ∞ as $t \rightarrow \infty$). The subcomplex $\Omega_0^*(M, t)$ spanned by eigenforms corresponding to the small and large eigenvalues of $\Delta(t)$ is finite dimensional. Under some mild conditions, it is shown that $(\Omega_0^*(M, t), d(t))$ converges to a geometric complex associated to the generalized Morse function as $t \rightarrow \infty$.

1 Introduction and Statement of Results

The purpose of this paper is to extend the Witten-Helffer-Sjöstrand theory (cf.[W],[H-S]) for a Morse function on compact manifold to a generalized Morse function. Such a generalized Morse function has all critical points either non-degenerate or of birth-death type, i.e. in some neighbourhood of the critical point and with respect to a convenient coordinate system, the function can be written

$$f(x_1, x_2, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + ax_n^3$$

for some $a \neq 0$.

The interest of generalized Morse function comes from the following theorem due to H.Chaltin(cf.[I]).

Theorem: If $\pi : E \rightarrow B$ is a smooth bundle with fibre a compact manifold M , then there exists $f : E \rightarrow \mathbb{R}$ so that for any $t \in B$

$$f_t = f|_{\pi^{-1}(t)} : M_t = \pi^{-1}(t) \rightarrow \mathbb{R}$$

is a generalized Morse function.

It is easy to see that in general one cannot have such a statement with f_t a Morse function.

Now, let us state the results of this paper. Let the eigenvalues of the operator

$$-\frac{d^2}{dx^2} + 9x^4 - 6x$$

be

$$0 < e_1 < e_2 \leq \dots \leq e_l \leq \dots$$

(See Lemma 2 in §2 for proof.)

Suppose M^n is a compact orientable Riemannian manifold, f be a generalized Morse function on M .

Suppose $x_1^k, \dots, x_{m_k}^k$ are all the non-degenerate critical points of f , of index k , $y_1^k, \dots, y_{m'_k}^k$ are all the critical points of birth-death type, of index k . Also, let $a_j^{(k)} \in \mathbb{R}$ be associated with y_j^k so that in some neighbourhood of y_j^k and with respect to a suitable oriented co-ordinate system,

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + a_j^{(k)} x_n^3$$

Suppose for simplicity that

$$|a_1^{(0)}| < |a_2^{(0)}| < \dots < |a_{m'_0}^{(0)}| < |a_1^{(1)}| < \dots < |a_{m'_1}^{(1)}| < \dots < |a_{m'_{n-1}}^{(n-1)}| \quad (1)$$

(in fact, the Witten-Helffer-Sjöstrand theory is very similar with minor modifications without assuming (1))

Also, let g be a Riemannian metric on M so that in the above co-ordinate system near the critical points x_j^k or y_j^k , g is the canonical metric on \mathbb{R}^n .

Consider the Witten deformation of the de Rham complex $(\Omega^*(M), d(t))$ with

$$d(t) = e^{-tf} d e^{tf} : \Omega^*(M) \rightarrow \Omega^*(M)$$

Consider the deformed Laplacian

$$\Delta(t) = d(t)d^*(t) + d^*(t)d(t)$$

When the above canonical coordinates near the critical points are used,

$$\Delta(t) = \Delta + t^2 |df|^2 + tA \quad (2)$$

where

$$A = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} [dx_i, i_{\partial_j}]$$

and i_{∂_j} denotes the contraction along the vector field ∂_j , dx_i is the exterior multiplication by the form dx_i and $[dx_i, i_{\partial_j}]$ denotes the commutator $dx_i i_{\partial_j} - i_{\partial_j} dx_i$.

There are two cases.

Case 1: x_j^k is non-degenerate.

$$f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 \quad (3)$$

$$|df|^2 = 4(x_1^2 + \dots + x_n^2)$$

$$A = -2 \sum_{i=1}^k [dx_i, i_{\partial_i}] + 2 \sum_{i=k+1}^n [dx_i, i_{\partial_i}] \quad (4)$$

Case 2: y_j^k is of birth-death type.

$$f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + a_j^{(k)} x_n^3 \quad (5)$$

$$|df|^2 = 4(x_1^2 + \dots + x_{n-1}^2) + 9(a_j^{(k)})^2 x_n^4$$

$$A = -2 \sum_{i=1}^k [dx_i, i_{\partial_i}] + 2 \sum_{i=k+1}^{n-1} [dx_i, i_{\partial_i}] + 6a_j^{(k)} x_n [dx_n, i_{\partial_n}] \quad (6)$$

For each critical point c , define the 'localized' operator $\overline{\Delta}_c(t) : C^\infty(\Lambda^*(R^n)) \rightarrow C^\infty(\Lambda^*(R^n))$ which is given by (2) where A is (4) if $c = x_j^k$ is nondegenerate, respectively (6) if $c = y_j^k$ is birth-death. The Laplace operator in (2) then is the canonical Laplace operator corresponding the canonical metric in R^n . The operator $\overline{\Delta}_c(t)$ then extends uniquely to a self-adjoint positive unbounded operator in $L^2(\Lambda^*(R^n))$.

Now suppose $\overline{\Delta}(t)$ is the 'localized' operator associated to a critical point of birth-death type. Since $L^2(\Lambda^*(R^n)) \cong L^2(\Lambda^*(R^{n-1})) \otimes L^2(\Lambda^*(R))$, $\overline{\Delta}(t)$ can be written as

$$\begin{aligned} \overline{\Delta}(t) &= \{ \Delta_{R^{n-1}} + 4t^2(x_1^2 + \dots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{\partial_i}] \} \otimes id \\ &\quad + id \otimes \{ \Delta_R + 9(a_j^{(k)})^2 t^2 x_n^4 + 6a_j^{(k)} t x_n [dx_n, i_{\partial_n}] \} \end{aligned}$$

where

$$\epsilon_i = \begin{cases} -1 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq n-1 \end{cases}$$

and $\Delta_{R^{n-1}}$, Δ_R are the Laplace operators on R^{n-1} and R respectively.

By Corollary in §2, $\Delta_R + 9(a_j^{(k)})^2 t^2 x_n^4 + 6a_j^{(k)} t x_n [dx_n, i_{\partial_n}] : L^2(\Lambda^*(R)) \rightarrow L^2(\Lambda^*(R))$ has discrete spectrum with eigenvalues

$$0 < e_1(|a_j^{(k)}| t)^{2/3} < e_2(|a_j^{(k)}| t)^{2/3} \leq e_3(|a_j^{(k)}| t)^{2/3} \leq \dots$$

Each eigenvalue has a multiplicity of 2 with corresponding eigenvectors consisting of a 0-form and a 1-form.

Let $\Delta^k(t) = \Delta(t) \mid_{L^2(\Lambda^k(M))}$.

Let $0 \leq E_1(t) \leq E_2(t) \leq \dots \leq E_l(t) \leq \dots$ be all the eigenvalues of $\Delta^k(t)$. Suppose for simplicity that

$$e_1 \mid a_{m'_{n-1}}^{(n-1)} \mid^{2/3} < e_2 \mid a_1^{(0)} \mid^{2/3}$$

This together with (1) imply that $e_1 \mid a_{m'_k}^{(k)} \mid^{2/3} < e_2 \mid a_1^{(k-1)} \mid^{2/3}$ for all k.

Theorem 1 (Quasi-classical limit of eigenvalues)

$$\begin{aligned} \lim_{t \rightarrow \infty} E_1(t) &= \dots = \lim_{t \rightarrow \infty} E_{m_k}(t) = 0 \\ \lim_{t \rightarrow \infty} \frac{E_{m_k+1}(t)}{t^{2/3}} &= e_1(\mid a_1^{(k-1)} \mid)^{2/3} < \lim_{t \rightarrow \infty} \frac{E_{m_k+2}(t)}{t^{2/3}} = e_1(\mid a_2^{(k-1)} \mid)^{2/3} < \dots \\ &\dots < \lim_{t \rightarrow \infty} \frac{E_{m_k+m'_{k-1}+m'_k}(t)}{t^{2/3}} = e_1(\mid a_{m'_k}^{(k)} \mid)^{2/3} \\ &\left(< \lim_{t \rightarrow \infty} \frac{E_{m_k+m'_{k-1}+m'_k+1}(t)}{t^{2/3}} = e_2(\mid a_1^{(k-1)} \mid)^{2/3} < \dots \right) \end{aligned}$$

Remarks:1. In fact, the eigenvectors corresponding to $E_1(t), \dots, E_{m_k}(t)$ are localized at the non-degenerate critical points of index k, while the eigenvectors corresponding to $E_{m_k+1}(t), \dots, E_{m_k+m'_{k-1}+m'_k}(t)$ are localized at the birth-death critical points of index k-1 and k. However, the eigenvectors are not necessarily localized at a single critical point.

2. If all the critical points of f are non-degenerate, then the above theorem should be formulated as follows (cf. [S] p219):

Theorem:

$$\begin{aligned} \lim_{t \rightarrow \infty} E_1(t) &= \dots = \lim_{t \rightarrow \infty} E_{m_k}(t) = 0 \\ 0 &< \lim_{t \rightarrow \infty} \frac{E_{m_k+1}(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{E_{m_k+2}(t)}{t} \leq \dots \end{aligned}$$

Let us index $a_1^{(0)}, \dots, a_{m'_0}^{(0)}, a_1^{(1)}, \dots, a_{m'_{n-1}}^{(n-1)}$ by b_1, \dots, b_N where $N = \sum_{k=0}^{n-1} m'_k$.

Also, for $0 \leq l \leq n-1, 1 \leq j \leq m'_l$ let

$$I_j^{(l)}(\epsilon) = [e_1(\mid a_j^{(l)} \mid)^{2/3} - \epsilon, e_1(\mid a_j^{(l)} \mid)^{2/3} + \epsilon]$$

Choose an ϵ small enough so that the family of intervals

$$\left\{ [0, \epsilon], I_j^{(l)}(\epsilon), [e_2(\mid a_1^{(0)} \mid)^{2/3} - \epsilon, \infty) \right\}_{0 \leq l \leq n-1, 1 \leq j \leq m'_l}$$

is pairwise disjoint. The pairwise disjointness is satisfied if ϵ is a positive number smaller than

$$\min_i \left(\frac{e_1}{2} |a_1^{(0)}|^{2/3}, \frac{e_1}{2} (|b_{i+1}|^{2/3} - |b_i|^{2/3}), \frac{1}{2} (e_2 |a_1^{(0)}|^{2/3} - e_1 |a_{m_{n-1}}^{(n-1)}|^{2/3}) \right)$$

As a consequence of Theorem 1, for t sufficiently large and ϵ satisfying the above disjointness condition, we have

$$\text{Spec}(t^{-2/3} \Delta^k(t)) \subset [0, \epsilon] \cup \left(\bigcup_{l=k-1, k; 1 \leq j \leq m_l'} I_j^{(l)}(\epsilon) \right) \cup [e_2(|a_1^{(0)}|)^{2/3} - \epsilon, \infty)$$

and

$$\begin{cases} \text{Card}(\text{Spec}(t^{-2/3} \Delta^k(t)) \cap [0, \epsilon]) = m_k \\ \text{Card}(\text{Spec}(t^{-2/3} \Delta^k(t)) \cap I_j^{(l)}(\epsilon)) = 1 \\ (for l = k-1, k; 1 \leq j \leq m_l') \end{cases}$$

Define

$$\Omega_{small}^k(M, t) = \text{Span}\{\psi(t) \in L^2(\Lambda^k(M)) | \Delta^k(t)\psi(t) = E(t)\psi(t), t^{-2/3}E(t) \in [0, \epsilon]\}$$

$$\begin{aligned} \Omega_{large, l, j}^k(M, t) &= \text{Span}\{\psi(t) \in L^2(\Lambda^k(M)) | \Delta^k(t)\psi(t) = E(t)\psi(t), \\ &\quad t^{-2/3}E(t) \in [e_1(|a_j^{(l)}|)^{2/3} - \epsilon, e_1(|a_j^{(l)}|)^{2/3} + \epsilon]\} \\ \Omega_{v, large}^k(M, t) &= \text{Span}\{\psi(t) \in L^2(\Lambda^k(M)) | \Delta^k(t)\psi(t) = E(t)\psi(t), \\ &\quad t^{-2/3}E(t) \in [e_2 - \epsilon, \infty)\} \end{aligned}$$

$$(\Omega_0^*(M, t), d(t)) = (\Omega_{small}^*(M, t), d(t)) \perp (\perp_{k, j} (\Omega_{large, k, j}^*(M, t), d(t)))$$

Corollary 1 $(\Omega^*(M), d(t))$ is equal to

$$(\Omega_{small}^*(M, t), d(t)) \perp (\perp_{k, j} (\Omega_{large, k, j}^*(M, t), d(t))) \perp (\Omega_{v, large}^*(M, t), d(t))$$

$(\Omega_{small}^*(M, t), d(t)), (\Omega_0^*(M, t), d(t))$ are complexes of finite dimensional vector spaces which calculate the de Rham cohomology of M .

Remark: Observe that $(\Omega_{large, k, j}^*(M, t), d(t))$ has dimension 2. It is spanned by a k -form and a $(k+1)$ -form localized at y_j^k (the localization of the forms at y_j^k is due to (1)).

As in the Helffer-Sjöstrand theory for a generic pair (f, g) , $(\Omega_0^*(M, t), d(t))$ converges as $t \rightarrow \infty$ to a geometric complex, which can be described as follows.

Let f be a self-indexing generalized Morse function, i.e.

$$\begin{cases} f(x_j^k) = k & \text{if } x_j^k \text{ is a non-degenerate critical point of index } k \\ f(y_j^k) \in (k, k+1) & \text{if } y_j^k \text{ is a birth-death critical point of index } k \end{cases}$$

Let $W_{x_j^k}^-$ be the descending manifold of a non-degenerate critical point x_j^k . For a birth-death critical point y_j^k , choose an open neighbourhood $U_{y_j^k}$ and a suitable co-ordinate s.t.

$$f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + a_j^{(k)} x_n^3$$

Let

$$\begin{aligned} W_{y_j^k}^{-,0} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap R^k \text{ for some } t \in R \\ &\quad \text{where } \gamma_x \text{ is the trajectory of } \text{Grad } f \text{ s.t. } \gamma_x(0) = x \} \\ W_{y_j^k}^{-,1} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap (R^k \times R_-) \text{ for some } t \in R \\ &\quad \text{where } R^k \times R_- = \{(x_1, \dots, x_k, 0, \dots, 0, x_n) \in R^n \mid x_n < 0\}\} \end{aligned}$$

when $a > 0$ with obvious modifications when $a < 0$. Then $W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1}$ are manifolds diffeomorphic to R^k, R^{k+1} respectively. Note that $W_{y_j^k}^{-,0} \cap W_{y_j^k}^{-,1} = \emptyset$

Define the descending manifold

$$W_{y_j^k}^- = W_{y_j^k}^{-,0} \cup W_{y_j^k}^{-,1}$$

which is then a manifold with boundary diffeomorphic to R_+^{k+1} . The ascending manifold $W_{y_j^k}^+$ is defined similarly.

Suppose the ascending and descending manifolds for any two critical points intersect transversally, then $\{W_{x_j^k}^-, W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1}\}$ form a CW-complex (see §3 for more details). While the incidence number between $W_{x_j^k}^-$ and $W_{x_{i+1}^k}^-$ is given by the intersection number between the ascending and descending manifolds in $f^{-1}(k + \frac{1}{2})$ i.e. between $W_{x_j^k}^+ \cap f^{-1}(k + \frac{1}{2})$ and $W_{x_{i+1}^k}^- \cap f^{-1}(k + \frac{1}{2})$, the incidence number is 1 between $W_{y_j^k}^{-,0}$ and $W_{y_j^k}^{-,1}$. However, those between $W_{x_j^k}^-$ and $W_{y_j^k}^{-,i}$ may be non-trivial ($i=0,1$). Let us denote the above described chain complex by $(C_*(M, f), \partial)$ (with $C_k(M, f) = \text{Span}\{W_{x_j^k}^-, W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1}\}$), its dual cochain complex by $(C^*(M, f), \delta)$.

Also, let us rescale the complex $(\Omega_0^*(M, t), d(t))$ to be

$$(\Omega_0^*(M, t), \tilde{d}(t)) = \left(\Omega_{small}^*(M, t), e^t \sqrt{\frac{\pi}{2t}} d(t) \right) \perp (\perp_{k,j} (\Omega_{large,j,k}^*(M, t), d(t)))$$

Theorem 2 *There exists $f^*(t) : (\Omega_0^*(M, t), \tilde{d}(t)) \rightarrow (C^*(M, f), \delta)$ which is a morphism of co-chain complexes s.t.*

$$f^*(t) = I + O(t^{-1})$$

w.r.t. some suitably chosen bases.

Definitions: 1.(i). Suppose x_j^k, x_i^{k+1} are two non-degenerate critical points, γ be a generalized trajectory between x_i^{k+1} and x_j^k , i.e. γ is a piecewise smooth curve with singularities at the birth-death points $y_1, \dots, y_{n(\gamma)}$ and

$$\gamma = \gamma_{x_i^{k+1}y_1} \cup \{y_1\} \cup \gamma_{y_1y_2} \cup \{y_2\} \cup \dots \cup \gamma_{y_{n(\gamma)}x_j^k}$$

where $\gamma_{y_l y_{l+1}}$ is a trajectory between y_l and y_{l+1} . Then one can associate $\epsilon = \pm 1$ to the trajectories $\gamma_{x_i^{k+1}y_1}, \gamma_{y_l y_{l+1}}, \gamma_{y_{n(\gamma)}x_j^k}$ as in the Witten-Morse theory. Then define

$$\epsilon_\gamma^{new} = (-1)^{n(\gamma)} \epsilon_{\gamma_{x_i^{k+1}y_1}} \left(\prod_{l=1}^{n(\gamma)-1} \epsilon_{\gamma_{y_l y_{l+1}}} \right) \epsilon_{\gamma_{y_{n(\gamma)}x_j^k}}$$

(ii) Suppose x_j^k is an non-degenerate critical point and y_i^k a birth-death critical point, γ be a generalized trajectory between them. With the above notation for γ and $y_1 = y_i^k$, define

$$\epsilon_\gamma^{new} = (-1)^{n(\gamma)} \left(\prod_{l=1}^{n(\gamma)-1} \epsilon_{\gamma_{y_l y_{l+1}}} \right) \epsilon_{\gamma_{y_{n(\gamma)}x_j^k}}$$

2. (i) The (generalized) incidence number between two critical points is defined as follows:

$$I(x_i^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_\gamma^{new}$$

$$I(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_\gamma^{new}$$

where γ is a generalized trajectory between the initial and end point.

(ii) Here we recall that the (ordinary) incidence number between two critical points(non-degenerate or birth-death) is

$$i(x_i^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_\gamma$$

$$i(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_\gamma$$

where γ is a trajectory between the two critical points.

Remark: Observe that in general

$$\epsilon_\gamma^{new} \neq \epsilon_\gamma$$

for a trajectory between two critical points. For example, if γ is a trajectory between a birth-death point y_i^k and a non-degenerate critical point x_j^k , then $\epsilon_\gamma^{new} = -\epsilon_\gamma$

With the above definition, we can reformulate Theorem 2 as follows:

Theorem 2': (Helffer-Sjöstrand) There exist orthonormal bases $\{E_{x_j^k}(t)\}$ of $\Omega_{small}^k(M, t)$, $\{E_{y_j^k}^0(t), E_{y_j^k}^1(t)\}$ of $\Omega_{large, k, j}^*(M, t)$ s.t.

$$\langle E_{x_i^{k+1}}(t), d(t)E_{x_j^k}(t) \rangle = e^{-t} \left(\sqrt{\frac{t}{\pi}} \sum_{\gamma} \epsilon_{\gamma}^{new} + O(t^{-1/2}) \right)$$

$$\langle E_{y_j^k}^1(t), d(t)E_{y_j^k}^0(t) \rangle = \sqrt{e_1}(a_j^k)^{1/3} t^{1/3} + O(t^{1/6})$$

$$\langle E_{y_{j_1}^k}^{i_1}(t), d(t)E_{y_{j_2}^k}^{i_2}(t) \rangle = 0 \text{ if } j_1 \neq j_2 \text{ for } t \text{ sufficiently large}$$

$$\langle E_{x_i^k}(t), d(t)E_{y_j^l}^{i'}(t) \rangle = \langle E_{y_j^l}^{i'}(t), d(t)E_{x_i^k}(t) \rangle = 0 \text{ for } t \text{ sufficiently large}$$

where $\sum_{\gamma} \epsilon_{\gamma}^{new} = I(x_i^{k+1}, x_j^k)$ is the incidence number between x_j^k and x_i^{k+1} defined above.

Inside the complex $(C^*(M, f), \delta)$, there is a subcomplex $(C_{nd}^*(M, f), \delta)$ such that

$$\dim C_{nd}^k(M, f) = m_k$$

where m_k is the number of non-degenerate critical points of index k . Note that this subcomplex is not generated by the non-degenerate critical points, since the latter in general do not generate a subcomplex. Instead the subcomplex is obtained by applying Lemma 3 repeatedly as is done in §3. See §3 for details.

Theorem 2'': $f^*(t) |_{\Omega_{small}^*(M, t)}: (\Omega_{small}^*(M, t), \tilde{d}(t)) \rightarrow (C^*(M, f), \delta)$ is an injective homomorphism of co-chain complexes whose image complex converges to $(C_{nd}^*(M, f), \delta)$ in $(C^*(M, f), \delta)$ as $t \rightarrow \infty$, more precisely,

$$f^k(t) \left(E_{x_j^k}(t) \right) = \hat{e}_{x_j^k} + O(t^{-1}) \text{ in } C^*(M, f)$$

where $\hat{e}_{x_j^k} = e_{x_j^k} + \sum_l I(y_l^k, x_j^k) e_{y_l^k}^0 \in C_{nd}^k(M, f)$.

Also, using similar consideration, one can extend the result to any representation $\rho: \pi_1(M) \rightarrow GL(V)$ (cf.[BZ]) or any representation $\rho: \pi_1(M) \rightarrow GL(W_A)$ where W_A is finite type Hilbert module over a finite von Neuman algebra A (cf.[BFSKM]).

The above question concerning the extension of Witten-Helffer- Sjöstrand theory for generalized Morse functions was raised in Dan Burghlelea's course on L_2 -topology. I would like to thank him for the problem and help in accomplishing this work. A parametrized version of the above theory will be presented in future work in collaboration with D. Burghlelea.

2 Witten Deformation for a Generalized Morse Function

Let f be a generalized Morse function on M^n , y be a critical point of birth-death type. Let (U_y, φ) be a chart s.t. $y \in U_y$, and

$$f(\varphi^{-1}x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + ax_n^3$$

Let g be a Riemannian metric on M s.t. $(\varphi^{-1})^*(g) = \delta_{ij}$.

Define

$$d(t) = e^{-tf} de^{tf}$$

$$\Delta(t) = d(t)d^*(t) + d^*(t)d(t)$$

Then in the coordinate system (U_y, φ) ,

$$\Delta(t) = \Delta + t^2 |df|^2 + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{\partial_i}] + 6ax_n t [dx_n, i_{\partial_n}]$$

where

$$|df|^2 = 4(x_1^2 + \dots + x_{n-1}^2) + 9a^2 x_n^4$$

$$\epsilon_i = \begin{cases} -1 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq n-1 \end{cases}$$

Define $\overline{\Delta}(t) : L^2(\Lambda^*(R^n)) \rightarrow L^2(\Lambda^*(R^n))$ to be given exactly by the above expression. Recall that

$$\begin{aligned} \overline{\Delta}(t) &= \{ \Delta_{R^{n-1}} + 4t^2(x_1^2 + \dots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{\partial_i}] \} \otimes id \\ &\quad id \otimes \{ \Delta_R + 9t^2 a^2 x_n^4 + 6atx_n [dx_n, i_{\partial_n}] \} \end{aligned}$$

Observe that the first term is exactly the Witten deformed Laplacian on $L^2(\Lambda^*(R^{n-1}))$ for the classical Morse theory (cf.[S],[W]) and that the two operators in parenthesis commute with each other. Hence to study the spectrum of $\overline{\Delta}(t)$, it suffices to find out the spectrum of $\Delta + 9t^2 a^2 x^4 + 6tax [dx, i_{\frac{d}{dx}}]$ on $L^2(\Lambda^*(R))$. Note that

$$[dx, i_{\frac{d}{dx}}](f) = -f$$

$$[dx, i_{\frac{d}{dx}}](f dx) = f dx$$

Therefore

$$\left\{ \Delta + 9t^2 a^2 x^4 + 6tax [dx, i_{\frac{d}{dx}}] \right\} (f) = \left(-\frac{d^2}{dx^2} + 9t^2 a^2 x^4 - 6tax \right) (f)$$

$$\left\{ \Delta + 9t^2 a^2 x^4 + 6atx \left[dx, i \frac{d}{dx} \right] \right\} (f dx) = \left(-\frac{d^2}{dx^2} + 9t^2 a^2 x^4 + 6atx \right) (f) dx$$

Define $R : L^2(R) \rightarrow L^2(R)$

$$(Rf)(x) = f(-x)$$

Then

$$R^{-1} \left(-\frac{d^2}{dx^2} + 9t^2 a^2 x^4 + 6atx \right) R = -\frac{d^2}{dx^2} + 9t^2 a^2 x^4 - 6atx$$

Hence, it is sufficient to consider

$$P(at) = -\frac{d^2}{dx^2} + 9t^2 a^2 x^4 - 6atx$$

where $P(t) \equiv -\frac{d^2}{dx^2} + 9t^2 x^4 - 6tx$ for $t \in \mathbb{R}$.

Define $U(\lambda) : L^2(R) \rightarrow L^2(R)$, $\lambda > 0$

$$(U(\lambda)f)(x) = \lambda^{1/2} f(\lambda x)$$

Lemma 1 For $t > 0$,

$$P(t) = U(t^{1/3}) \left(t^{2/3} P(1) \right) U(t^{-1/3})$$

Lemma 2 (i) $P(1)$ has compact resolvent, hence has discrete spectrum.

$$0 \leq e_1 \leq e_2 \leq \dots \leq e_l \leq \dots$$

(ii) The smallest eigenvalue of $P(1)$ is strictly positive and is simple, i.e.

$$0 < e_1 < e_2 \leq \dots$$

(iii) Let Ξ_1 be a normalized eigenfunction of $P(1)$ corresponding to the smallest eigenvalue e_1 . Then one can choose Ξ_1 so that $\Xi_1(x) > 0$ for all $x \in \mathbb{R}$. In particular $\Xi_1(0)^{-1}$ exists.

Corollary: $P(at)$ has spectrum

$$0 < e_1(|at|)^{2/3} < e_2(|at|)^{2/3} \leq e_3(|at|)^{2/3} \leq \dots \leq e_l(|at|)^{2/3} \leq \dots$$

Proof of Lemma 2: (i) $P(1)$ has compact resolvent because $V(x) = 9x^4 - 6x \rightarrow \infty$ as $|x| \rightarrow \infty$ and is bounded from below. (cf. [RS] p249) It is positive because it is the restriction of the deformed Laplacian associated with the function x^3 on the invariant subspace $L^2(R)$.

(ii) Let $\left(-\frac{d^2}{dx^2} + 9x^4 - 6x\right)(f) = 0$

Then

$$f = c_1 e^{-x^3} + c_2 e^{-x^3} \int_0^x e^{2u^3} du$$

One checks that $f \in L^2(R)$ iff $c_1 = c_2 = 0$. This shows $e_1 > 0$.

To show that e_1 is simple, one apply the Feynman-Kac formula to show that $e^{-P(1)}$ has a strictly positive kernel. This implies that the largest eigenvalue of $e^{-P(1)}$ is simple, hence the simplicity of the smallest eigenvalue of $P(1)$. (cf. [GJ] §3.3)

(iii) The application of Feynman-Kac formula in (ii) shows that Ξ_1 can be chosen s.t. $\Xi_1(x) > 0$ almost everywhere. One shows that actually $\Xi_1(x) > 0$ for all $x \in R$. \square

Now, we have shown that the smallest eigenvalue of $P(at)$ is $e_1(|at|)^{2/3} > 0$ and is simple. Since $-\frac{d^2}{dx^2} + 9a^2t^2x^4 + 6atx$ is conjugated to $P(at)$ by an isometry R , the same is true for its smallest eigenvalue. Hence $-\frac{d^2}{dx^2} + 9a^2t^2x^4 + 6atx$ on $L^2(\Lambda^*(R))$ has smallest eigenvalue $e_1(|at|)^{2/3}$ of multiplicity 2, the corresponding eigenvectors are $\Xi_1(x) (\in \Omega^0(R))$ and $\Xi_1(-x)dx (\in \Omega^1(R))$.

Returning to the 'localized' operator,

$$\begin{aligned} \overline{\Delta}(t) = & \left\{ \Delta_{R^{n-1}} + 4t^2(x_1^2 + \dots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{\partial_i}] \right\} \otimes id \\ & + id \otimes \{ \Delta_R + 9a^2t^2x_n^4 + 6atx_n [dx_n, i_{\partial_n}] \} \end{aligned}$$

The smallest eigenvalue is also $e_1(|at|)^{2/3}$ and of multiplicity 2, whose eigenvectors are spanned by a k -form and a $(k+1)$ -form.

$$\omega_k(t) = t^{(n-1)/4+1/6} \xi_1(t^{1/2}x_1) \dots \xi_1(t^{1/2}x_{n-1}) \Xi_1(t^{1/3}x_n) dx_1 \wedge \dots \wedge dx_k$$

$$\omega_{k+1}(t) = t^{(n-1)/4+1/6} \xi_1(t^{1/2}x_1) \dots \xi_1(t^{1/2}x_{n-1}) \Xi_1(-t^{1/3}x_n) dx_1 \wedge \dots \wedge dx_k \wedge dx_n$$

where $\xi_1(x)$ is the *groundstate* of $-\frac{d^2}{dx^2} + 4x^2$ i.e. $\xi_1(x) = e^{-tx^2}$.

With the above observations, the proof of Theorem 1 follows essentially the arguments in [S](pp 219-222). See Appendix for sketch of proof.

3 Helffer-Sjöstrand Theory for a Generalized Morse Function

Definition: A pair (f, g) is said to satisfy the Morse-Smale condition where f is a generalized Morse function if for any two critical points x and y , the ascending manifold W_x^+ and the descending manifold W_y^- , w.r.t. $-Grad_g f$, intersect transversally.

In the case of a birth-death critical point y_j^k of index k such that (5) holds, define

$$\begin{aligned} W_{y_j^k}^{+,0} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap R^{n-k-1} \text{ for some } t \in R \\ &\quad \text{where } R^{n-k-1} = \{(0, \dots, 0, x_{k+1}, \dots, x_{n-1}, 0) \in R^n\}\} \\ W_{y_j^k}^{+,1} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap (R^{n-k-1} \times R_+ \text{ for some } t \in R \\ &\quad \text{where } R^{n-k-1} \times R_+ = \{(0, \dots, 0, x_{k+1}, \dots, x_n) \in R^n \mid x_n > 0\}\} \end{aligned}$$

while $W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1}$ are defined similarly as in §1. Then the ascending and descending manifolds are defined as follows:

$$\begin{aligned} W_{y_j^k}^+ &= W_{y_j^k}^{+,0} \cup W_{y_j^k}^{+,1} \\ W_{y_j^k}^- &= W_{y_j^k}^{-,0} \cup W_{y_j^k}^{-,1} \end{aligned}$$

Proposition 1 *For any pair (f, g) , there is a C^1 approximation g' such that $g = g'$ in a neighbourhood of the critical points of f and (f, g') satisfies the Morse-Smale condition.*

Proof: The proof is the same as in [Sm].

Definition: Let f be a generalized Morse function. f is said to be self-indexing if

$$\begin{cases} f(x_j^k) = k & \text{if } x_j^k \text{ is a non-degenerate critical point of index } k \\ f(y_j^k) \in (k, k+1) & \text{if } y_j^k \text{ is birth-death critical point of index } k \end{cases}$$

Proposition 2 *For any generalized Morse function f , there exists a self-indexing generalized Morse function f' such that f and f' have the same critical points and corresponding indexes.*

Proof: The proof is similar as in [M] §4.

Definition: A pair (f, g) is called a generalized triangulation if

(i) f is a self-indexing generalized Morse function on M and in a neighbourhood U_c of any critical point c , one can introduce local coordinates s.t. $g = \delta_{ij}$ and

(a) if c is non-degenerate

$$f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

(b) if c is of birth-death type

$$f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{n-1}^2 + ax_n^3$$

(ii) (f,g) satisfy the Morse-Smale condition.

Let f be a generalized Morse function, $W_{x_j^k}^-$ be the descending manifold of a non-degenerate critical point. For a birth-death critical point y_j^k , recall

$$\begin{aligned} W_{y_j^k}^{-,0} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap R^k \text{ for some } t \in R \\ &\quad \text{where } \gamma_x \text{ is the trajectory of } \text{Grad } f \text{ s.t. } \gamma_x(0) = x \} \\ W_{y_j^k}^{-,1} &= \{x \in M \mid \gamma_x(t) \in U_{y_j^k} \cap (R^k \times R_-) \text{ for some } t \in R \\ &\quad \text{where } R^k \times R_- = \{(x_1, \dots, x_k, 0, \dots, 0, x_n) \in R^n \mid x_n < 0\}\} \end{aligned}$$

where $R^k = \{(x_1, \dots, x_k, 0, \dots, 0) \in R^n\}$.

Then we have

Theorem: Suppose (f,g) is a generalized triangulation, then

- (i) $\left\{ W_{x_j^k}^-, W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1} \right\}_{0 \leq k \leq n; 1 \leq j \leq m_k \text{ or } m'_k}$ is a CW-complex.
- (ii) Let $(C_*(M, f), \partial)$ be the cellular chain complex of the above CW-complex (as described in §1), $(C^*(M, f), \delta)$ be its dual co-chain complex. Then $\text{Int} : (\Omega^*(M), d) \rightarrow (C^*(M, f), \delta)$

$$\omega \mapsto \int_W \omega$$

is a morphism of co-chain complexes.

Proof: A proof of this theorem in the case of a Morse function can be found in [L]. The same argument also works in the case of a generalized Morse function. However, a better argument for this is the following:

(i) One first verify that the partition $\left\{ W_{x_j^k}^-, W_{y_j^k}^{-,0}, W_{y_j^k}^{-,1} \right\}$ is a stratification in the sense of Whitney (see [V2] for definition). Using the tubular neighbourhood theorem (Proposition 2.6 [V1]) and the fact that each stratum is diffeomorphic

with an Euclidean space, one concludes that this partition is a CW-complex.

(ii) The fact that integration is well defined and represents a morphism of cochain complexes follows from Stokes theorem in the framework of integration theory on stratified sets (cf.[F],[V2]). \square

As a consequence, the composition

$$(\Omega_0^*(M, f, t), d(t)) \xrightarrow{e^{tf}} (\Omega^*(M), d) \xrightarrow{\text{Int}} (C^*(M, f), \delta)$$

is also a morphism of co-chain complexes.

Let

$$M_{x_j^k} = M \setminus [(\cup_{l \neq j} B(x_l^k, \eta)) \cup (\cup_l B(y_l^k, \eta))]$$

Let $\Delta_{M_{x_j^k}}(t)$ be the corresponding Laplace operator on $M_{x_j^k}$ with Dirichlet boundary condition, $\Psi_{x_j^k}(t)$ be an eigenvector corresponding to the smallest eigenvalue of $\Delta_{M_{x_j^k}}^k(t)$ of norm one.

Similarly, let

$$M_{y_j^k} = M \setminus [(\cup_l B(x_l^k, \eta)) \cup (\cup_{l \neq j} B(y_l^k, \eta))]$$

With $\Delta_{M_{y_j^k}}(t)$ similarly defined, let $\Psi_{y_j^k}^0(t)$, respectively $\Psi_{y_j^k}^1(t)$ be the smallest eigenvector of $\Delta_{M_{y_j^k}}^k(t)$, $\Delta_{M_{y_j^k}}^{k+1}(t)$ of norm one.

Define $J_k(t) : C^k(M, f) \longrightarrow \Omega^k(M)$ by

$$J_k(t) \left(e_{x_j^k} \right) = \Psi_{x_j^k}(t)$$

$$J_{k+i}(t) \left(e_{y_j^k}^i \right) = \Psi_{y_j^k}^i(t), \quad i = 0, 1$$

where $\{e_{x_j^k}, e_{y_j^k}^i\}$ is the dual basis of $\{W_{x_j^k}, W_{y_j^k}^i\}$.

Let $Q_{k,small}(t)$, $Q_{k+i,k,j}(t)$ be the orthogonal projection onto $\Omega_{small}^k(M, t)$ and $\Omega_{large,k,j}^{k+i}(M, t)$ respectively.

Define $Q_k(t) : C^k(M, f) \longrightarrow \Omega_0^k(M, t)$ by

$$Q_k(t) \left(\Psi_{x_j^k}(t) \right) = Q_{k,small}(t) \left(\Psi_{x_j^k}(t) \right)$$

$$Q_{k+i}(t) \left(\Psi_{y_j^k}^i(t) \right) = Q_{k+i,k,j}(t) \left(\Psi_{y_j^k}^i(t) \right)$$

Let

$$H_k(t) = (Q_k(t)J_k(t))^* (Q_k(t)J_k(t))$$

$$\tilde{J}_k(t) = Q_k(t)J_k(t)(H_k(t))^{-1/2}$$

Then $\tilde{J}_k(t) : C^k(M, f) \rightarrow \Omega_0^k(M, t)$ is an isometry.

Define $E_{x_j^k}(t) = \tilde{J}_k(t) \left(e_{x_j^k} \right)$, $E_{y_j^k}^i(t) = \tilde{J}_k(t) \left(e_{y_j^k}^i \right)$.

Note that $E_{x_j^k}(t) \in \Omega_{small}^k(M, t)$, $E_{y_j^k}^i(t) \in \Omega_{large,k,j}^{k+i}(M, t)$.

Proposition 3

$$E_{x_j^k}(t) = \left(\frac{2t}{\pi} \right)^{n/4} e^{-tx^2} (dx_1 \wedge \dots \wedge dx_k + O(t^{-1}))$$

$$E_{y_j^k}^0(t) = \left(\frac{2t}{\pi} \right)^{\frac{n-1}{4}} e^{-t(x_1^2 + \dots + x_{n-1}^2)} (|at|)^{1/6} \Xi_1((at)^{1/3} x_n) (dx_1 \wedge \dots \wedge dx_k + O(t^{-1}))$$

on $U_{x_j^k}$ and $U_{y_j^k}$ respectively.

Remark: Note that $(\frac{2t}{\pi})^{\frac{n-1}{4}}$ and $|at|^{1/6}$ are the normalization constants for $e^{-t(x_1^2+\dots+x_{n-1}^2)}$ and $\Xi_1((at)^{1/3}x_n)$ respectively, i.e.

$$\|(\frac{2t}{\pi})^{\frac{n-1}{4}} e^{-t(x_1^2+\dots+x_{n-1}^2)}\| = \| |at|^{1/6} \Xi_1((at)^{1/3}x)\| = 1$$

Proof: One can follow the argument in [HS] or [BZ]. Note that the term $(|at|)^{-1/6}\Xi_1((at)^{1/3}x)$ is the ground state of $-\frac{d^2}{dx^2} + 9a^2t^2x^4 - 6atx$. So it is also the first term of the asymptotic expansion. \square

Recall that we have defined ϵ_γ^{new} for a generalized trajectory between two critical points and the incidence number $I(x, y)$ between two critical points. See §1 for definitions. Also define $Int_k = Int|_{\Omega^k(M)}$. With these definitions, we have

Proposition 4 (i)

$$Int_k e^{tf} \left(E_{x_j^k}(t) \right) = \left(\frac{2t}{\pi} \right)^{\frac{n-2k}{4}} e^{tk} \left(e_{x_j^k} + \sum_l I(y_l^k, x_j^k) e_{y_l^k}^0 + O(t^{-1}) \right)$$

(ii)

$$Int_k e^{tf} \left(E_{y_j^k}^0(t) \right) = \left(\frac{2t}{\pi} \right)^{\frac{n-1-2k}{4}} e^{f(y_j^k)} \Xi_1(0) |a_j^{(k)} t|^{1/6} \left(e_{y_j^k}^0 + \sum_{l \neq j} \beta_{lj}(t) e_{y_l^k}^0 + O(t^{-1}) \right)$$

(iii)

$$Int_{k+1} e^{tf} \left(E_{y_j^k}^1(t) \right) = \left(\frac{2t}{\pi} \right)^{\frac{n-1-2k}{4}} e^{tf(y_j^k)} \frac{\Xi_1(0) |a_j^{(k)} t|^{1/6}}{\sqrt{e_1} |a_j^{(k)} t|^{1/3}} \left(\delta \left(e_{y_j^k}^0 + \sum_{l \neq j} \beta_{lj}(t) e_{y_l^k}^0 \right) + O(t^{-1}) \right)$$

Proof: We introduce the following notations. Let $y_1^k, \dots, y_{m_k}^k$ be all the birth-death critical points of index k. Let

$$f(y_1^k) = \dots = f(y_{r_1}^k) < f(y_{r_1+1}^k) = \dots = f(y_{r_1+r_2}^k) < f(y_{r_1+r_2+1}^k) < \dots < \dots < f(y_{r_1+\dots+r_{l_k-1}+1}^k) = \dots = f(y_{r_1+\dots+r_{l_k}}^k)$$

where $m_k = r_1 + \dots + r_{l_k}$.

Also, for $1 \leq q \leq l_k$ let

$$L_q^{(k)} = \{y_l^k \mid r_1 + \dots + r_{q-1} + 1 \leq l \leq r_1 + \dots + r_{q-1} + r_q\}$$

(i) It is clear from [HS],[BZ] that

$$\int_{W_{x_j^k}^-} e^{tf} E_{x_j^k}(t) = \left(\frac{2t}{\pi} \right)^{\frac{n-2k}{4}} e^{tk} (1 + O(t^{-1}))$$

Suppose $y_l^k \in L_1^{(k)}$, let

$$\partial W_{y_l^k}^{-,1} = W_{y_l^k}^{-,0} + i(y_l^k, x_j^k) W_{x_j^k}^- + \left(\sum_{i \neq j} i(y_l^k, x_i^k) W_{x_i^k}^- + \sum_i i(y_l^k, y_i^{k-1}) W_{y_i^{k-1}}^{-,1} \right)$$

where $i(y_l^k, x_j^k)$ is the (ordinary) incidence number between y_l^k and x_j^k defined in §1. Denote the expression inside the parenthesis by R , the remainder term.

Therefore,

$$\int_{\partial W_{y_l^k}^{-,1}} e^{tf} E_{x_j^k}(t) = \int_{W_{y_l^k}^{-,0}} e^{tf} E_{x_j^k}(t) + i(y_l^k, x_j^k) \int_{W_{x_j^k}^-} e^{tf} E_{x_j^k}(t) + \int_R e^{tf} E_{x_j^k}(t)$$

But by Stoke's Theorem,

$$\int_{\partial W_{y_l^k}^{-,1}} e^{tf} E_{x_j^k}(t) = \int_{W_{y_l^k}^{-,1}} e^{tf} (d(t) E_{x_j^k}(t)) = \int_{W_{y_l^k}^{-,1}} e^{tf} \left(\sum_i \lambda_i(t) E_{x_i^{k+1}}(t) \right)$$

for some exponentially decaying functions $\lambda_i(t)$. Since $|E_{x_i^{k+1}}(t)(x)|$ decreases as $e^{-t|f(x)-f(x_i^{k+1})|}$, $\int_{\partial W_{y_l^k}^{-,1}} e^{tf} E_{x_j^k}(t)$ is of smaller order compared with $\int_{W_{x_j^k}^-} e^{tf} E_{x_j^k}(t)$. The same is true for $\int_R e^{tf} E_{x_j^k}(t)$.

Hence,

$$\begin{aligned} \int_{W_{y_l^k}^{-,0}} e^{tf} E_{x_j^k}(t) &= -i(y_l^k, x_j^k) \int_{W_{x_j^k}^-} e^{tf} E_{x_j^k}(t) + O(t^{-1}) \\ &= I(y_l^k, x_j^k) \left(\frac{2t}{\pi} \right)^{\frac{n-2k}{4}} e^{tk} + O(t^{-1}) \end{aligned}$$

One can show by using finite induction on q that for any $y_l^k \in L_q^{(k)}$

$$\int_{W_{y_l^k}^{-,0}} e^{tf} E_{x_j^k}(t) = I(y_l^k, x_j^k) \left(\frac{2t}{\pi} \right)^{\frac{n-2k}{4}} e^{tk} + O(t^{-1})$$

This proves (i).

(ii) A direct computation shows that

$$\int_{W_{y_j^k}^{-,0}} e^{tf} E_{y_j^k}^0(t) = \left(\frac{2t}{\pi} \right)^{\frac{n-1-2k}{4}} e^{tf(y_j^k)} \Xi_1(0) |a_j^{(k)} t|^{1/6} (1 + O(t^{-1}))$$

Next note that by choosing a coordinate system $(x_1^{(j)}, \dots, x_k^{(j)})$ on $W_{y_j^k}^{-,0}$ and extending it to a neighbourhood of $W_{y_j^k}^+ = W_{y_j^k}^{+,0} \cup W_{y_j^k}^{+,1}$, one can show as in [HS] p 276-8 that if $x \neq y_l^k$,

$$E_{y_j^k}^0(t) = \left(\frac{2t}{\pi} \right)^{\frac{n-1}{4}} |at|^{1/6} e^{-td(x, y_j^k)} \left(dx_1^{(j)} \wedge \dots \wedge dx_k^{(j)} + O(t^{-1}) \right)$$

where $d(x, y_j^k)$ is the Agmon distance between x and y_j^k , i.e. w.r.t. the metric $|df|^2 dg$, but it is not necessarily true for $x = y_l^k$.

So we let

$$E_{y_j^k}^0(t)(y_l^k) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}} |at|^{1/6} e^{-td(y_l^k, y_j^k)} c_{lj}(t)$$

where $c_{lj}(t) \in \Lambda^k(T_{y_l^k}(M))$. By [HS],

$$|c_{lj}(t)| = O(e^{\epsilon t}) \text{ for any } \epsilon > 0$$

For $x \in U_{y_l^k} \cap W_{y_l^k}^{-,0}$,

$$f(x) = f(y_l^k) - (x_1^{(l)})^2 - \dots - (x_k^{(l)})^2$$

Therefore,

$$\int_{W_{y_l^k}^{-,0}} e^{tf} E_{y_j^k}^0(t) = e^{tf(y_l^k)} \int_{U_{y_l^k} \cap W_{y_l^k}^{-,0}} e^{-t((x_1^{(l)})^2 + \dots + (x_k^{(l)})^2)} E_{y_j^k}^0(t) + \text{lower order terms}$$

By the stationary phase approximation formula ([D] p23-4), we have

$$\begin{aligned} & \int_{W_{y_l^k}^{-,0}} e^{tf} E_{y_j^k}^0(t) \\ &= e^{tf(y_l^k)} \left(\frac{2t}{\pi}\right)^{\frac{n-1-2k}{4}} |at|^{\frac{1}{6}} e^{-td(y_l^k, y_j^k)} c_{lj}(t) \left(\partial_{x_1^{(l)}}, \dots, \partial_{x_k^{(l)}}\right) + \text{lower order terms} \\ &= \left(\frac{2t}{\pi}\right)^{\frac{n-1-2k}{4}} |at|^{1/6} e^{tf(y_j^k)} \Xi_1(0) \beta_{lj}(t) \end{aligned}$$

since $d(y_l^k, y_j^k) = f(y_l^k) - f(y_j^k)$ and for some $\beta_{lj}(t)$. Hence (ii) is proved.

Note that (cf.[HS] p265, [HS1] p138)

$$|\beta_{lj}(t)| = O(e^{\epsilon t}) \text{ for any } \epsilon > 0$$

(iii) Since $d(t)E_{y_j^k}^0(t) = \lambda(t)E_{y_j^k}^1(t)$ for some $\lambda(t) \neq 0$ when t is sufficiently large (this follows from (7) below) and

$$Int_{k+1} \left(e^{tf} d(t) E_{y_j^k}^0(t) \right) = \delta \left(Int_k e^{tf} E_{y_j^k}^0(t) \right)$$

by (ii), we have

$$Int_{k+1} e^{tf} \left(E_{y_j^k}^1 \right) = \left(\frac{2t}{\pi}\right)^{\frac{n-1-2k}{4}} e^{tf(y_j^k)} \frac{\Xi_1(0)(|at|)^{1/6}}{\lambda(t)} \left(\delta(e_{y_j^k}^0 + \sum_{l \neq j} \beta_{lj}(t) e_{y_l^k}^0) + O(t^{-1}) \right)$$

(iii) is proved by noting that

$$\lambda(t) = \sqrt{e_1} |at|^{1/2} + \text{lower order terms} \quad \square$$

Using Proposition 4(i), we can prove

Theorem 2'': (Helffer-Sjöstrand) There exist orthonormal bases $\{E_{x_j^k}(t)\}$ of $\Omega_{small}^k(M, t)$, $\{E_{y_j^k}^0(t), E_{y_j^k}^1(t)\}$ of $\Omega_{large, k, j}^*(M, t)$ s.t.

$$\langle E_{x_i^{k+1}}(t), d(t)E_{x_j^k}(t) \rangle = e^{-t} \left(\sqrt{\frac{t}{\pi}} \sum_{\gamma} \epsilon_{\gamma}^{new} + O(t^{-1/2}) \right)$$

$$\langle E_{y_j^k}^1(t), d(t)E_{y_j^k}^0(t) \rangle = \sqrt{e_1} (a_j^{(k)} t)^{1/3} + O(t^{1/6})$$

$$\langle E_{y_{j_1}^{k_1}}^{i_1}(t), d(t)E_{y_{j_2}^{k_2}}^{i_2}(t) \rangle = 0 \text{ if } j_1 \neq j_2 \text{ for } t \text{ sufficiently large}$$

$$\langle E_{x_i^k}(t), d(t)E_{y_j^k}^{i'}(t) \rangle = \langle E_{y_j^k}^{i'}(t), d(t)E_{x_i^k}(t) \rangle = 0 \text{ for } t \text{ sufficiently large}$$

where $\sum_{\gamma} \epsilon_{\gamma}^{new} = I(x_i^{k+1}, x_j^k)$ is the incidence number between x_i^{k+1} and x_j^k defined in §1.

Proof: We prove the first and second equalities, the others are obvious.

Let $d(t)E_{x_j^k}(t) = \sum_i \lambda_{ij}(t)E_{x_i^{k+1}}(t)$ for some $\lambda_{ij}(t)$.

Since

$$Int_{k+1} e^{tf} (d(t)E_{x_j^k}(t)) = \delta Int_k e^{tf} (E_{x_j^k}(t))$$

By Proposition 4(i), we have

$$\begin{aligned} & \sum_i \lambda_{ij}(t) \left(\frac{2t}{\pi} \right)^{\frac{n-2k-2}{4}} e^{t(k+1)} \left(e_{x_i^{k+1}} + \sum_l I(y_l^{k+1}, x_i^{k+1}) e_{y_l^{k+1}}^0 + O(t^{-1}) \right) \\ &= \left(\frac{2t}{\pi} \right)^{\frac{n-2k}{4}} \left(\delta e_{x_j^k} + \sum_l I(y_l^k, x_j^k) \delta e_{y_l^k}^0 + O(t^{-1}) \right) \end{aligned}$$

By comparing the coefficients of $e_{x_i^{k+1}}$,

$$\lambda_{ij}(t) \left(\frac{2t}{\pi} \right)^{-1/2} e^t = i(x_i^{k+1}, x_j^k) + \sum_l I(y_l^k, x_j^k) i(x_i^{k+1}, y_l^k) + O(t^{-1})$$

But by definition of $I(x, y)$,

$$I(x_i^{k+1}, x_j^k) = i(x_i^{k+1}, x_j^k) + \sum_l I(y_l^k, x_j^k) i(x_i^{k+1}, y_l^k) = \sum_{\gamma} \epsilon_{\gamma}^{new}$$

Hence the first equality is proved. For the second equality, note that $E_{y_j^k}^i(t)$ are normalized eigenforms in $\Omega_{large, k, j}^*(M, t)$ and

$$\|d(t)E_{y_j^k}^0(t)\|^2 = \langle e_{y_j^k}^0(t), \Delta^k(t)E_{y_j^k}^0(t) \rangle = e_1 |a_j^{(k)} t|^{2/3} + \text{lower order terms} \quad (7)$$

So,

$$\langle e_{y_j^k}^1(t), d(t)E_{y_j^k}^0(t) \rangle = \pm \sqrt{e_1}(|a_j^{(k)}| t)^{1/3} + \text{lower order terms } \square$$

In view of Proposition 4, define $f^k(t) : \Omega_0^k(M, t) \rightarrow C^k(M, f)$ s.t.

$$\begin{aligned} f^k(t) \left(E_{x_j^k}(t) \right) &= \left(\frac{\pi}{2t} \right)^{\frac{n-2k}{4}} e^{-tk} \text{Int}_k e^{tf} \left(E_{x_j^k}(t) \right) \\ f^k(t) \left(E_{y_j^k}^0(t) \right) &= \text{Int}_k e^{tf} \left(E_{y_j^k}^0(t) \right) \\ f^{k+1}(t) \left(E_{y_j^k}^1(t) \right) &= \text{Int}_{k+1} e^{tf} \left(E_{y_j^k}^1(t) \right) \end{aligned}$$

Let

$$\begin{aligned} \left(\Omega_0^*(M, t), \tilde{d}(t) \right) &= \left(\Omega_{small}^*(M, t), e^t(\pi/2t)^{1/2} d(t) \right) \\ &\perp \left(\perp_{k,j} \left(\Omega_{large,k,j}^*(M, t), d(t) \right) \right) \end{aligned}$$

Also define

$$\begin{aligned} \hat{e}_{x_j^k} &= e_{x_j^k} + \sum_l I(y_l^k, x_j^k) e_{y_l^k}^0 \\ \hat{e}_{y_j^k}^0 &= e_{y_j^k}^0 \\ \hat{e}_{y_j^k}^1 &= \delta(e_{y_j^k}^0) \end{aligned}$$

Then we have

Proposition 5 $f^*(t) : (\Omega_0^*(M, t), \tilde{d}(t)) \rightarrow (C^*(M, f), \delta)$ is a morphism of co-chain complexes s.t.

$$f^k(t) \left(E_{x_j^k}(t) \right) = \hat{e}_{x_j^k} + O(t^{-1})$$

with $f^k(t) \left(E_{y_j^k}^0(t) \right), f^{k+1} \left(E_{y_j^k}^1(t) \right)$ given by (ii) and (iii) in Proposition 4.

Let the matrix associated to the linear map $f^k(t)$ w.r.t. the bases

$$\left\{ E_{x_j^k}(t), E_{y_j^k}^0(t), E_{y_j^{k-1}}^1(t) \right\} \text{ and } \left\{ \hat{e}_{x_j^k}, e_{y_j^k}^0, e_{y_j^{k-1}}^1 \right\}$$

be

$$F^k(t) = \begin{pmatrix} I & O(e^{kt}) & N_1^k(t) \\ O(t^{-1}) & M^k(t) & N_2^k(t) \\ O(t^{-1}) & O(e^{kt}) & N_3^k(t) \end{pmatrix}$$

where $O(t^{-1})$ in a certain entry of the matrix means that the corresponding entry is of the order $O(t^{-1})$. Here $M^k(t)$ is $\left(\frac{2t}{\pi} \right)^{\frac{n-1-2k}{4}} \Xi_1(0)$ times the following

matrix

$$\begin{pmatrix} |a_1^{(k)} t|^{\frac{1}{6}} e^{tf(y_1^k)} & |a_2^{(k)} t|^{\frac{1}{6}} e^{t(f(y_1^k) - \epsilon_0)} \beta_{12}(t) & \cdots & |a_{m_k'}^{(k)} t|^{\frac{1}{6}} e^{t(f(y_1^k) - \epsilon_0)} \beta_{1m_k'}(t) \\ |a_1^{(k)} t|^{\frac{1}{6}} e^{tf(y_1^k)} \beta_{21}(t) & |a_2^{(k)} t|^{\frac{1}{6}} e^{tf(y_2^k)} & \cdots & |a_{m_k'}^{(k)} t|^{\frac{1}{6}} e^{t(f(y_2^k) - \epsilon_0)} \beta_{2m_k'}(t) \\ \vdots & \vdots & \ddots & \vdots \\ |a_1^{(k)} t|^{\frac{1}{6}} e^{tf(y_1^k)} \beta_{m_k'1}(t) & |a_2^{(k)} t|^{\frac{1}{6}} e^{tf(y_2^k)} \beta_{m_k'2}(t) & \cdots & |a_{m_k'}^{(k)} t|^{\frac{1}{6}} e^{tf(y_{m_k'}^k)} \end{pmatrix}$$

Note that we have used the fact that the birth-death points are indexed such that

$$f(y_1^k) \leq f(y_2^k) \leq \dots \leq f(y_{m_k'}^k)$$

so that the above matrix is approximately lower triangular. Hence, $M^k(t)$ is invertible for sufficiently large t .

Also, let

$$A^k(t) = \text{diag} \left((\sqrt{e_1} |a_1^{(k)} t|^{1/3})^{-1}, \dots, (\sqrt{e_1} |a_{m_k'}^{(k)} t|^{1/3})^{-1} \right)$$

Define for $1 \leq j \leq m_k'$,

$$\hat{E}_{y_j^k}^0(t) = (M^k(t))^{-1} \left(E_{y_j^k}^0(t) \right)$$

$$\hat{E}_{y_j^k}^1(t) = (M^k(t) A^k(t))^{-1} E_{y_j^k}^1(t)$$

Observe that $\{\hat{E}_{y_j^k}^i(t)\}$ is approximately orthogonal whose elements are still localized at the corresponding birth-death points.

Let

$$B^k(t) = \begin{pmatrix} I & 0 & 0 \\ 0 & (M^k(t))^{-1} & 0 \\ 0 & 0 & (M^{k-1}(t) A^{k-1}(t))^{-1} \end{pmatrix}$$

Then the matrix associated to $f^k(t)$ w.r.t. the new bases

$$\{E_{x_j^k}(t), \hat{E}_{y_j^k}^0(t), \hat{E}_{y_j^k}^1(t)\} \text{ and } \{\hat{e}_{x_j^k}, e_{y_j^k}^0, e_{y_j^{k-1}}^1\}$$

is

$$F^k(t) B^k(t) = \begin{pmatrix} I & O(e^{kt}) & N_1^k(t) \\ O(t^{-1}) & M^k(t) & N_2^k(t) \\ O(t^{-1}) & O(e^{kt}) & N_3^k(t) \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & (M^k(t))^{-1} & 0 \\ 0 & 0 & (M^{k-1}(t) A^{k-1}(t))^{-1} \end{pmatrix}$$

Suppose that

$$\delta^{(k-1)} = \begin{pmatrix} \delta_{11}^{(k-1)} & \delta_{12}^{(k-1)} & \delta_{13}^{(k-1)} \\ \delta_{21}^{(k-1)} & \delta_{22}^{(k-1)} & \delta_{23}^{(k-1)} \\ \delta_{31}^{(k-1)} & \delta_{32}^{(k-1)} & \delta_{33}^{(k-1)} \end{pmatrix}$$

w.r.t. the bases $\{\hat{e}_{x_j^k}, e_{y_j^k}^0, e_{y_j^k}^1\}$.

Then,

$$F^k(t)B^k(t) = \begin{pmatrix} I & O(t^{-1}) & \delta_{13}^{(k-1)} + O(t^{-1}) \\ O(t^{-1}) & I & \delta_{23}^{(k-1)} + O(t^{-1}) \\ O(t^{-1}) & O(t^{-1}) & \delta_{33}^{(k-1)} + O(t^{-1}) \end{pmatrix}$$

To see this, it suffices to show

$$Int_k e^{tf} \left(\hat{E}_{y_j^{k-1}}^1(t) \right) = \delta(e_{y_j^{k-1}}^0) + O(t^{-1}) \quad (8)$$

But by Proposition 4(iii),

$$Int_k e^{tf} \left(E_{y_j^{k-1}}^1 \right) = \delta \left(\sum_l (M^{k-1}(t)A^{k-1}(t))_{lj} e_{y_l^{k-1}}^0 + O(t^{-1}) \right)$$

Using the definition of $\hat{E}_{y_j^{k-1}}^1(t)$, (8) follows.

Hence, with the definition of $\hat{e}_{y_j^k}^i$ on p18, we finally have

Theorem 2: $f^*(t) : (\Omega_0^*(M, t), \tilde{d}(t)) \longrightarrow (C^*(M, f), \delta)$ is a morphism of cochain complexes s.t.

$$f^*(t) = I + O(t^{-1})$$

w.r.t. the bases $\{E_{x_j^k}(t), \hat{E}_{y_j^k}^i(t)\}$ and $\{\hat{e}_{x_j^k}, \hat{e}_{y_j^k}^i\}$.

Inside $(C^*(M, f), \delta)$, there is a subcomplex $(C_{nd}^*(M, f), \delta)$ such that

$$\dim C_{nd}^k(M, f) = m_k$$

where m_k is the number of non-degenerate critical points of index k . This subcomplex can be obtained by application of the following Lemma.

Lemma 3 Suppose (C^*, δ) is a cochain complex such that

$$C^* = \begin{cases} \tilde{C}^* & \text{if } * \neq k, k+1 \\ \tilde{C}^* \oplus R & \text{if } * = k \text{ or } k+1 \end{cases}$$

Let $\dim(\tilde{C}^q) = n_q$, for $0 \leq q \leq n$,

$$\{e_{x_1^q}, \dots, e_{x_{n_q}^q}\} \text{ be a basis of } \tilde{C}^q$$

so that

$$\{e_{x_1^k}, \dots, e_{x_{n_k}^k}, e_y^0\} \text{ is a basis of } \tilde{C}^k \oplus R$$

and

$$\{e_{x_1^{k+1}}, \dots, e_{x_{n_{k+1}}^{k+1}}, e_y^1\} \text{ is a basis of } \tilde{C}^{k+1} \oplus R$$

and w.r.t. the above bases,

$$\delta^{(k)} = \begin{pmatrix} i(x_1^{k+1}, x_1^k) & \cdots & i(x_1^{k+1}, x_{n_k}^k) & i(x_1^{k+1}, y) \\ \vdots & \ddots & \vdots & \vdots \\ i(x_{n_{k+1}}^{k+1}, x_1^k) & \cdots & i(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & i(x_{n_{k+1}}^{k+1}, y) \\ i(y, x_1^k) & \cdots & i(y, x_{n_k}^k) & 1 \end{pmatrix}$$

Then with the following change of bases in C^k and C^{k+1} ,

$$\begin{aligned} \{e_{x_l^k}, e_y^0\}_{1 \leq l \leq n_k} &\longmapsto \{e_{x_l^k} - i(y, x_l^k)e_y^0, e_y^0\}_{1 \leq l \leq n_k} \\ \{e_{x_l^{k+1}}, e_y^1\}_{1 \leq l \leq n_{k+1}} &\longmapsto \left\{ e_{x_l^{k+1}}, \delta e_y^0 = e_y^1 + \sum_{j=1}^{n_{k+1}} i(x_j^{k+1}, y)e_{x_j^{k+1}} \right\}_{1 \leq l \leq n_{k+1}} \end{aligned}$$

we have

(i)

$$\delta^{(k)} = \begin{pmatrix} i'(x_1^{k+1}, x_1^k) & \cdots & i'(x_1^{k+1}, x_{n_k}^k) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ i'(x_{n_{k+1}}^{k+1}, x_1^k) & \cdots & i'(x_{n_{k+1}}^{k+1}, x_{n_k}^k) & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $i'(x_i^{k+1}, x_j^k) = i(x_i^{k+1}, x_j^k) - i(x_i^{k+1}, y)i(y, x_j^k)$.

(ii)

$$\begin{aligned} \delta^{(k-1)} &= \begin{pmatrix} i(x_1^k, x_1^{k-1}) & \cdots & i(x_1^k, x_{n_{k-1}}^{k-1}) \\ \vdots & \ddots & \vdots \\ i(x_{n_k}^k, x_1^{k-1}) & \cdots & i(x_{n_k}^k, x_{n_{k-1}}^{k-1}) \\ 0 & \cdots & 0 \end{pmatrix} \\ \delta^{(k+1)} &= \begin{pmatrix} i(x_1^{k+2}, x_1^{k+1}) & \cdots & i(x_1^{k+2}, x_{n_{k+1}}^{k+1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ i(x_{n_{k+2}}^{k+2}, x_1^{k+1}) & \cdots & i(x_{n_{k+2}}^{k+2}, x_{n_{k+1}}^{k+1}) & 0 \end{pmatrix} \end{aligned}$$

Corollary: Let

$$(C')^* = \begin{cases} C^* & \text{if } * \neq k \\ \text{Span}\{e_{x_l^k} - i(y, x_l^k)e_y^0\}_{1 \leq l \leq n_k} & \text{if } * = k \end{cases}$$

$$(C'')^* = \begin{cases} 0 & \text{if } * \neq k, k+1 \\ e_y^0 & \text{if } * = k \\ \delta e_y^0 & \text{if } * = k+1 \end{cases}$$

Then

$$(C^*, \delta) = ((C')^*, \delta) \oplus ((C'')^*, \delta)$$

In particular, $((C')^*, \delta)$ is a subcomplex of (C^*, δ) . Both of them calculate the same cohomology.

Remark: For the application of Lemma 3, it is clear that \tilde{C}^* need not be generated only by cells corresponding to non-degenerate critical points, but it can also be generated by cells corresponding to birth-death points. Let

$$k < f(y_1^k) \leq \dots \leq f(y_{m_k}^k) < k+1$$

We 'eliminate' y_1^k first by applying Lemma 3 and obtain a subcomplex $((C^{(1)})^*, \delta)$. Note that

$$\begin{cases} e_{y_2^k}^0 = e_{y_2^k}^0 - i(y_1^k, y_2^k) e_{y_1^k}^0 \in ((C^{(1)})^k \\ e_{y_2^k}^1 \in (C^{(1)})^{k+1} \\ \delta(e_{y_2^k}^0) = e_{y_2^k}^1 + \dots \end{cases}$$

Therefore, the assumptions in Lemma 3 are satisfied and we can apply Lemma 3 to 'eliminate' y_2^k and obtain $((C^{(2)})^*, \delta)$. Hence by applying Lemma 3 repeatedly, $(C_{nd}^*(M, f), \delta)$ is obtained.

Proof of Lemma 3: The Lemma can be proved by direct calculation.

Theorem 2'': $f^*(t) |_{\Omega_{small}^*(M, t)}: (\Omega_{small}^*(M, t), \tilde{d}(t)) \longrightarrow (C^*(M, f), \delta)$ is an injective homomorphism of cochain complexes whose image complex converges to $(C_{nd}^*(M, f), \delta)$ in $(C^*(M, f), \delta)$ as $t \rightarrow \infty$, more precisely,

$$f^k(f) \left(E_{x_j^k}(t) \right) = \hat{e}_{x_j^k} + O(t^{-1}) \text{ in } C^*(M, f)$$

Remark: One can show by induction that

$$\hat{e}_{x_j^k} = e_{x_j^k} + \sum_l I(y_l^k, x_j^k) e_{y_l^k}^0 \in C_{nd}^k(M, f)$$

Appendix

Sketch of Proof (of Theorem 1):

Let C_{bd} be the set of birth-death critical points of f ,

C_{nd} be the set of non-degenerate critical points of f ,

$0 = e_1^k = \dots = e_{m_k}^k$ be the smallest eigenvalues of $\bigoplus_{j \in C_{nd}} \overline{\Delta_j^k}(1)$

$0 \leq e_{m_k+1}^k \leq e_{m_k+2}^k \leq \dots$ be the eigenvalues of $\bigoplus_{j \in C_{bd}} \overline{\Delta_j^k}(1)$

Let $\{\Psi_l^k(1)\}_l^\infty$ be the eigenvectors corresponding to e_l^k of $\bigoplus_{j \in Crit(f)} \overline{\Delta_j^k}(1)$

More generally, let $\{\Psi_l^k(t)\}_l^\infty$ be the eigenvectors corresponding to $e_l^k t^{2/3}$ of $\bigoplus_{j \in Crit(f)} \overline{\Delta_j^k}(t)$.
 Note that

$$\Psi_l^k(t) = \begin{cases} U(t^{1/2})\Psi_l^k(1) & \text{if } 1 \leq l \leq m_k \\ U(t^{1/3})\Psi_l^k(1) & \text{if } m_k + 1 \leq l \end{cases}$$

Then Theorem 1 is essentially equivalent to

$$\lim_{t \rightarrow \infty} \frac{E_l(t)}{t^{2/3}} = e_l^k$$

The proof is divided into 2 steps.

$$(i) \lim_{t \rightarrow \infty} \frac{E_l(t)}{t^{2/3}} \leq e_l^k$$

This follows by similar arguments as in [S]. Let $\{J_j\}_{j \in Crit(f) \cup \{0\}}$ be a partition of unity on M (cf.[S] p 27). Let $\Psi_l^k(t)$ be an eigenvector of $\overline{\Delta_{j(l)}^k}(t)$, define $\varphi_l^k(t) = J_{j(l)}\Psi_l^k(t)$. Then $\{\varphi_l^k(t)\}$ form a set of approximate eigenvectors in $L^2(\Lambda^k(M))$ with

$$\langle \varphi_m(t), \Delta^k(t)\varphi_n(t) \rangle = e_n^k t^{2/3} \delta_{nm} + o(t^{2/3})$$

by using the minimax principle, (i) follows.

$$(ii) \lim_{t \rightarrow \infty} \frac{E_l(t)}{t^{2/3}} \geq e_l^k$$

To prove (ii), one has to modify slightly the arguments in [S]. It suffices to construct, for $e \in (e_l^k, e_{l+1}^k)$, a symmetric operator $R(t)$ of rank l s.t.

$$\Delta^k(t) \geq t^{2/3}e + R(t) + o(t^{2/3}) \dots (*)$$

To construct $R(t)$ define $\Delta_j^k(t) : C^\infty(\Lambda^k(M)) \rightarrow C^\infty(\Lambda^k(M))$

$$\Delta_j^k(t) = \Delta^k + t^2 f_j + tA$$

$$f_j(x) = \begin{cases} |df(x)|^2 & \text{if } x \in U_j \\ > 0 & \text{if } x \notin U_j \end{cases}$$

Let $0 \leq E_1^{(j)}(t) \leq E_2^{(j)}(t) \leq \dots \leq E_l^{(j)}(t) \leq \dots$ be the eigenvalues of $\Delta_j^k(t)$,

$\Psi_1^{(j)}(t), \psi_2^{(j)}(t), \dots, \Psi_l^{(j)}(t), \dots$ be the corresponding eigenvectors of $\Delta_j^k(t)$.

For $j \in C_{bd}$, let $0 \leq e_1^{(j)} \leq e_2^{(j)} \leq \dots \leq e_j^{(j)} \leq \dots$ be the eigenvalues of $\overline{\Delta_j^k}(1)$, then one can show that

$$\lim_{t \rightarrow \infty} \frac{E_l^{(j)}(t)}{t^{2/3}} = e_l^{(j)}$$

Define n_j s.t. $e_{n_j}^{(j)} < e < e_{n_j+1}^{(j)}$

$$P_j(t) = \begin{cases} \text{orthogonal projection onto } \text{span} \left\{ \Psi_l^{(j)} \right\}_{1 \leq l \leq n_j} & \text{if } j \in C_{bd} \\ \text{orthogonal projection onto smallest eigenvector of } \Delta_j^k(t) & \text{if } j \in C_{nd} \end{cases}$$

$$R_j(t) = (\Delta_j^k(t) - t^{2/3}e) P_j(t)$$

Define

$$R(t) = \sum_{j \in \text{Crit}(f)} J_j R_j(t) J_j$$

which is a symmetric operator of rank l .

To verify (*), observe that by IMS localization formula (cf.[S] p28)

$$\Delta^k(t) \geq t^{2/3}eJ_0^2 + \sum_{j \in \text{Crit}(f)} J_j \Delta_j^k(t) J_j + O(1)$$

Then (*) follows from the definition of $R(t)$.

One finishes the proof by showing that

$$\lim_{t \rightarrow \infty} E_1(t) = \dots = \lim_{t \rightarrow \infty} E_{m_k}(t) = 0 \quad \square$$

References

- [BZ] Bismut,J. and Zhang,W.: *An extension of a theorem by Cheeger and Müller, Asterisque 205(1992).*
- [BFKM] Burghelea,D.,Friedlander,L.,Kappeler,T. and McDonald,P.: *Analytic and Reidemeister torsion for representations in finite type Hilbert modules, preprint. 1994*
- [D] Duistermaat,J: *Fourier integral operator, New York University,1973.*
- [F] Ferrarotti,M.: *Some results about integration on regular stratified sets, Annali di Mat. Pura ed Appl. 150 (1988) 263-279.*
- [GJ] Glimm,J and Jaffe,A.: *Quantum physics, a functional integral point of view, 2nd edition, Springer-Verlag 1987.*
- [HS] Helffer,B. and Sjöstrand,J.: *Puits multiples en mecanique semi-classique,IV Etude du complexe de Witten. Comm. in PDE 10 (1985),245-340.*
- [HS1] Helffer,B. and Sjöstrand,J.: *Puits multiples en limite semi-classique II-interaction moleculaire-symetries-perturbation, Ann. Inst. Henri Poincare (Section physique theorique) Vol 42, No 2, (1985),127-212.*
- [I] Igusa,K.: *Parametrized Morse theory and its applications, Proc. of ICM (1990) 643-652.*
- [L] Laudенbach,F.: *Appendix in [BZ].*
- [M] Milnor,J.: *Lectures on the h-cobordism theorem, Princeton Univ. Press (1965).*

- [RS] Reed,M. and Simon,B.: *Methods of modern mathematical physics IV: analysis of operators*, Acad Press 1978.
- [S] Simon,B.: *Schrödinger operators with application to quantum mechanics and global geometry*, Springer-Verlag 1987.
- [Sm] Smale,S.: *On gradient dynamical systems*, *Annals of Math.* 74 No.1 (1961) 199-206.
- [V1] Verona,A.: *Homological properties of abstract prestratification*, *Rev. Roum. Math. Pures et Appl.* XVII No.7 p1109-1121(1972).
- [V2] Verona,A.: *Integration on Whitney prestratification*, *Rev. Roum. Pures et Appl.*XVII No.9 (1972).
- [W] Witten,E.: *Supersymmetry and Morse theory*, *J. of Diff. Geom.* 17 (1982) 661-692.